Math 1552

Section 8.3:
Powers and Products of Trigonometric Functions

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Today's Goal:

- Use trigonometric formulas to reduce more difficult integrals until we can perform a *u*-substitution.

- Idea: rewrite the function in terms of just one trig function after "breaking off" its derivative for a *u*-substitution

Useful Trig Identities

$$(*)\sin^2 x + \cos^2 x = 1$$

$$(*)1 + \tan^2 x = \sec^2 x$$

$$(*)\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$$

$$(*)\cos^2 x = \frac{1}{2}[1 + \cos(2x)]$$

seen this one before

$$(*)\sin(2x) = 2\sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

if you remember this

$$\tan^2 x + 1 = \sec^2 x \quad (1)$$

$$1 + \cot^2 x = \csc^2 x \quad (2)$$

(Where do these come from?)

(1) $(\sin^2 x + \cos^2 x = 1) * \frac{1}{\cos^2 x}$

(2) $* \frac{1}{\sin^2 x}$

Special cases: x=at, y=bt

<u>Example 1.1</u>: Evaluate the following integral: $\int \tan^3(x)dx = I$

$\tan^2 x + 1 = \sec^2 x$

<u>idea</u>: use that $\frac{d}{dx}[\tan x] = \sec^2 x$ OR

$\frac{d}{dx}[\sec x] = \tan x \cdot \sec x$,

then perform a u-sub: either $u = \tan x$ OR

$u = \sec x$

$$\tan^3 x = \tan x \cdot \tan^2 x = \tan x(\sec^2 x - 1)$$

$$\rightarrow I = \underbrace{\int \tan x \cdot \sec^2 x \, dx}_{I_1} - \underbrace{\int \tan x \, dx}_{I_2}$$

To evaluate I_1: use a u-sub

$$u = \tan x, \quad du = \sec^2 x$$

$$I_1 = \int u \, du = \frac{u^2}{2} + C_1 = \frac{\tan^2 x}{2} + C_1$$

To evaluate I_2: $\tan x = \frac{\sin x}{\cos x} \longrightarrow$ u-sub

(we have seen this before)

$$I_2 = \ln|\sec x| + C_2$$

In total:

$$I = \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

$$= \frac{\tan^2 x}{2} + \ln|\cos x| + C$$

Example 1.2: Evaluate the following integral: $I = \int \cos^2(x)\cot(x)dx$

$$\cos^2 x \cdot \cot x = \frac{\cos^3 x}{\sin x}, \quad \cos^2 x = 1 - \sin^2 x$$

$$= \frac{\cos x (1 - \sin^2 x)}{\sin x}$$

$$I = \int \frac{(1 - \sin^2 x)}{\sin x} \cdot \cos x \, dx$$

→ use a u-sub: $u = \sin x$

$du = \cos x \, dx$

$$I = \int \left(\frac{1}{u} - u\right) du$$

$$= \ln|u| - \frac{u^2}{2} + C$$

$$= \ln|\sin x| - \frac{\sin^2 x}{2} + C$$

Example 1.3: Evaluate the following integral: $I = \int \sin^4(x)dx$

$$\sin^2 x = \frac{1}{2}[1-\cos(2x)]$$

$$\cos^2 x = \frac{1}{2}[1+\cos(2x)]$$

$$\sin^4 x = \sin^2 x \cdot \sin^2 x$$
$$= \frac{1}{4}(1-\cos(2x))^2 \quad (*)$$

Intuition: we know how to integrate

$$\int \cos(\alpha x)\,dx = -\frac{1}{\alpha}\sin(\alpha x) + C$$

$$(*) = \frac{1}{4}\left(1 - 2\cos(2x) + \cos^2(2x)\right)$$

$$= \frac{1}{4}\left(1 - 2\cos(2x) + \frac{1}{2}(1 + \cos(4x))\right)$$

$$= \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$$

So $I = \frac{3}{8}\int dx - \frac{1}{2}\int \cos(2x)\,dx$

$$+ \frac{1}{8}\int \cos(4x)\,dx$$

$$= \frac{3x}{8} + \frac{1}{4}\sin(2x) - \frac{1}{32}\sin(4x) + C$$

Example 2.1: Evaluate. $\int \tan^3(x)\sec^3(x)dx = I$

recall: $\tan^2 x + 1 = \sec^2 x$

$$\tan^3 x \cdot \sec^3 x = \tan x\,(\sec^2 x - 1)\cdot \sec^3 x$$
$$= (\tan x \cdot \sec x)(\sec^4 x - \sec^2 x)$$

$$I = \int \sec^4 x\,(\tan x \cdot \sec x)\,dx$$
$$- \int \sec^2 x\,(\tan x \cdot \sec x)\,dx$$

$\rightarrow$ use a u-sub: $u = \sec x$

$du = \tan x \cdot \sec x \, dx$

$$I = \int (u^4 - u^2)\, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

Example 2.2: Evaluate: $I = \int \sec^3(x)\,dx$

what can we try:

① $\sec^3 x = \sec x(\tan^2 x + 1)$

$= \underline{\sec x \cdot \tan^2 x} + \sec x$

× we do not know how to integrate this

② IBP! $\int u\,dv = uv - \int v\,du$

$$u = \sec x \qquad dv = \sec^2 x \, dx$$

$$du = \tan x \cdot \sec x \, dx \qquad v = \tan x$$

$$I = \sec x \cdot \tan x - \underbrace{\int \tan^2 x \cdot \sec x \, dx}_{\tan^2 x = \sec^2 x - 1}$$

$$I = \sec x \cdot \tan x - I + \int \sec x \, dx$$

$$\Rightarrow I = \frac{1}{2}\left(\sec x \cdot \tan x + \ln\left|\sec x + \tan x\right|\right) + C$$

Evaluate the integral.

$I = \int \sin^2(x)\cos^3(x)dx$

$(A)\ \frac{1}{5}\sin^5(x)+C$

$\boxed{(B)\ \frac{1}{3}\sin^3(x)-\frac{1}{5}\sin^5(x)+C}$

$(C)\ \frac{1}{12}\sin^3(x)\cos^4(x)+C$

$(D)\ -\frac{1}{3}\cos^3(x)+\frac{1}{5}\cos^5(x)+C$

apply: $\sin^2 x+\cos^2 x=1$

$\cos^3 x = \cos x\,(1-\sin^2 x)$

$I = \int (\sin^2 x - \sin^4 x)\cdot \cos x\,dx$

$\longrightarrow$ use a u-sub:

$u=\sin x,\ du=\cos x\,dx$

$I = \int (u^2-u^4)\,du = \frac{u^3}{3}-\frac{u^5}{5}+C$

Extra Problem: Evaluate the integral. $\int \frac{\sec^4(4x)}{\tan^9(4x)}dx = I$

apply: $\cdot \sec^2 x = \tan^2 x + 1$

$\cdot \frac{d}{dx}\left[\tan(4x)\right] = 4\sec^2(4x)$

Write: $\frac{\sec^4(4x)}{\tan^9(4x)} = \sec^2(4x)\frac{(\tan^2(4x)+1)}{\tan^9(4x)}$

$\longrightarrow$ perform a u-sub: $u = \tan(4x)$

$$\frac{1}{4}du = \sec^2(4x)\,dx$$

$\longleftrightarrow du = 4\sec^2(4x)\,dx$

$$I = \frac{1}{4}\int (u^{-7} + u^{-9})\,du$$

$$= \frac{1}{4}\left(\frac{u^{-6}}{-6} + \frac{u^{-8}}{-8}\right) + C$$

$$= -\left(\frac{1}{24\tan^6(4x)} + \frac{1}{32\tan^8(4x)}\right) + C$$

Extra problem: Evaluate the integral. $\int \sin(5x)\cos(3x)dx = I$

Hint: $\sin(5x)\cos(3x) = \frac{1}{2}(\sin(2x) + \sin(8x))$

$$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

$(*) \mapsto$ "x" $= 5x$, $y = 3x$ shows how to get the hint

$$I = \frac{1}{2}\int \sin(2x)dx + \frac{1}{2}\int \sin(8x)dx$$

$$= -\frac{1}{4}\cdot\cos(2x) - \frac{1}{16}\cos(8x) + C$$

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Section 8.4:

Trigonometric Substitution

NOT covered on quiz 2

"trig subs"

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Today's Learning Goals

- Identify which types of integrals can be solved with a trigonometric substitution (a.k.a., trig subs)
- Learn which substitution matches which general form
- Evaluate integrals using the method of trigonometric substitution

Trigonometric Substitutions

recall the challenge problem:

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$

We use a trig substitution when no other integration method will work, and when the integral contains one of these terms:

$$a^2 - x^2$$

OR $$x^2 - a^2$$

OR $$a^2 + x^2$$

Rules to Trig Substitutions

- Begin by replacing x with a trig function.

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- Don’t forget to also replace *dx* with the appropriate trig function.

(important)

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- Use trig identities to solve the resulting integral.

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- Begin by replacing *x* with a trig function.
- Don’t forget to also replace *dx* with the appropriate trig function.
- Use trig identities to solve the resulting integral.
- Be sure to rewrite your final answer in terms of *x*. (*)
- *Know how to derive the corresponding right triangle in each of the three cases we consider below without memorizing them*